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On the Boundary Barycentric Calculus

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In this paper we shall prove that any semicontinuous affine real function, defined on a compact convex set, satisfies the boundary barycentric calculus, with respect to the measure induced on the extremal boundary of the set by any maximal Radon probability measure. © 1988 Academic Press, Inc.

I

Let K be any non-empty compact convex subset of a Hausdorff locally convex topological real vector space E. We shall denote by ex K the extremal boundary of K; i.e., the set of the extreme points of K.

We shall denote by $\mathscr{M}^1_+(K)$ the convex set of all Radon probability measures on K; it is compact in the topology $\sigma(C(K)^*; C(K))$, when considered as a subset of $C(K)^*$, the Banach dual space of the C^*-algebra $C(K)$ of all continuous complex functions on K.

We shall consider on $\mathscr{M}^1_+(K)$ the order relation, due to Choquet and Meyer, and defined as

$$\mu \prec \nu \text{ iff } \mu(\varphi) \leqslant \nu(\varphi), \qquad \forall \varphi \in S(K),$$

where $S(K)$ is the set of all convex continuous real functions on K. It is inductive (see [8, 12, 15]); hence, any $\mu \in \mathscr{M}^1_+(K)$ is majorized by a maximal $\nu \in \mathscr{M}^1_+(K)$, by Zorn's Lemma.

Any $\mu \in \mathscr{M}^1_+(K)$ has a barycenter $b(\mu) \in K$, uniquely determined by the condition

$$h(b(\mu)) = \int_K h(x)\, d\mu(x), \qquad \forall h \in A(K),$$

where $A(K) = S(K) \cap (-S(K))$ is the set of all affine continuous real functions on K.

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Remark. $b(\mu)$ is the integral with respect to μ of the continuous vector function $K \ni x \mapsto x \in E$, defined on K (see [16, Theorem 3.27, p. 74]).

It is obvious that $\mu \prec \nu \Rightarrow b(\mu) = b(\nu)$; it follows that any $x \in K$ is represented by a maximal $\mu \in \mathscr{M}^1_+(K)$: just start with the Dirac measure ε_x at x, and choose a maximal $\mu \in \mathscr{M}^1_+(K)$, such that $\varepsilon_x \prec \mu$.

For any $x \in K$, we shall denote by $\mathscr{M}^1_x(K)$ the compact convex subset of $\mathscr{M}^1_+(K)$, consisting of all $\mu \in \mathscr{M}^1_+(K)$, such that $b(\mu) = x$.

II

If K is metrizable, then ex K is a G_δ-subset of K, and any maximal measure $\mu \in \mathscr{M}^1_+(K)$ is *concentrated* on ex K; i.e.,

$$\mu(\operatorname{ex} K) = 1;$$

(see [7, 8, 12, 15, 17]).

In the general, possibly non-separable case, it may happen that ex K is not measurable, and then, according to the Choquet–Meyer Theorem, any maximal measure $\mu \in \mathscr{M}^1_+(K)$ is *pseudoconcentrated* on ex K; i.e., $\mu(D) = 0$, for any $D \in \mathscr{B}_0(K)$, such that $D \cap (\operatorname{ex} K) = \varnothing$ (see [12, 15]). Here $\mathscr{B}_0(K)$ denotes the σ-algebra of the Baire measurable subsets of K; i.e., the smallest σ-algebra of subsets of K, containing all closed G_δ-subsets of K. By $\mathscr{B}(K)$ we shall denote the σ-algebra of the Borel measurable subsets of K: it is the smallest σ-algebra of subsets of K, containing all closed subsets of K.

In general, for any topological space (T, τ), we can consider the σ-algebra $\mathscr{B}_0(T; \tau)$ of all Baire measurable subsets of (T, τ), i.e., the smallest σ-algebra of subsets of T, containing all τ-closed G_δ-subsets of T. We shall also consider the σ-algebra $\mathscr{B}(T; \tau)$ of all Borel measurable subsets of (T, τ), i.e., the smallest σ-algebra of subsets of T, containing all τ-closed subsets of T.

III

It is easy to see that the Choquet–Meyer Theorem can be stated also as follows: for any maximal measure $\mu \in \mathscr{M}^1_+(K)$, the outer measure $\mu_0^*(\operatorname{ex} K) = 1$. Here, and in the following, $\mu_0 = \mu \mid \mathscr{B}_0(K)$ is the restriction of μ to $\mathscr{B}_0(K)$.

It follows that by the formula

$$\tilde{\mu}_0(D \cap (\operatorname{ex} K)) = \mu(D), \qquad D \in \mathscr{B}_0(K),$$

one can define correctly a probability measure $\tilde{\mu}_0$ on the σ-algebra $\tilde{\mathscr{B}}_0(\text{ex } K)$ of subsets of ex K, given by

$$\tilde{\mathscr{B}}_0(\text{ex } K) = \{D \cap (\text{ex } K); D \in \mathscr{B}_0(K)\}.$$

Unfortunately, $\tilde{\mathscr{B}}_0(K)$ does not seem, in general, to be the σ-algebra of the Baire measurable, or the Borel measurable, subsets of ex K, corresponding to a suitable topology on ex K.

Besides the topology on ex K, induced by that of K, several other topologies have been considered on ex K (see [1, 3, 9]). One of the best is the so-called *Choquet topology*, introduced by Boboc and Bucur (see [4]), and defined as follows.

A subset $F \subset K$ is said to be *extremal* if its characteristic function χ_F is convex. Then the set $\mathscr{F}(K)$ of all compact extremal subsets of K is the set of all closed subsets for a topology on K. The Choquet topology on ex K is the topology induced on ex K by this topology; i.e., the set

$$\tilde{\mathscr{F}}(\text{ex } K) = \{F \cap (\text{ex } K); F \in \mathscr{F}(K)\}$$

is the set of all closed subsets of ex K in the Choquet topology.

Denoting the Choquet topology by C, we recall that (ex K, C) is a T_1-quasi-compact topological space; it is Hausdorff if, and only if, ex K is closed in K, in which case C coincides with the topology induced on ex K by the original topology of K (see [5]).

As shown by Batty (see [2, 3]; and also [18, 19]), any maximal measure $\mu \in \mathscr{M}^1_+(K)$ induces a measure $\tilde{\mu}$ on $\mathscr{B}(\text{ex } K; C)$, which is *regular*, in the sense that

$$\tilde{\mu}(\tilde{B}) = \sup\{\tilde{\mu}(\tilde{F}); \tilde{F} \in \tilde{\mathscr{F}}(\text{ex } K), \tilde{F} \subset \tilde{B}\},$$

for any $\tilde{B} \in \mathscr{B}(\text{ex } K; C)$; and, moreover, we have

$$\tilde{\mathscr{B}}_0(\text{ex } K) \subset \mathscr{B}(\text{ex } K; C)_{\tilde{\mu}}^{\sim},$$

where in the right hand member we have the Lebesgue completion of $\mathscr{B}(\text{ex } K; C)$ with respect to $\tilde{\mu}$. Maintaining the notation $\tilde{\mu}$ for the Lebesgue extension of $\tilde{\mu}$, we also have

$$\tilde{\mu}_0 = \tilde{\mu} \mid \tilde{\mathscr{B}}_0(\text{ex } K).$$

IV

Let h_0, h_1: $K \to \mathbb{R}$ be two bounded affine functions, such that $h_0(x) \leqslant h_1(x)$, $x \in K$. We shall consider the convex set

$$K_0 = \{(x, t); x \in K, h_0(x) \leqslant t \leqslant h_1(x)\} \subset E \times \mathbb{R}; \tag{IV.1}$$

we shall denote by $p\colon K_0 \to K$ and $q\colon K_0 \to \mathbb{R}$ the restrictions of the corresponding projections, and by $\Gamma(h_0)$ and $\Gamma(h_1)$ the graphs of the given functions

$$\Gamma(h_i) = \{(x, h_i(x)); x \in K\} \subset E \times \mathbb{R},$$

for $i = 0, 1$. We have the following

LEMMA 1. $\operatorname{ex} K_0 = \{(x, h_0(x)); x \in \operatorname{ex} K\} \cup \{(x, h_1(x)); x \in \operatorname{ex} K\}$.

Proof. Easy application of the definitions.

The following lemma is almost obvious.

LEMMA 2. *If h_0 and h_1 are, moreover, continuous, then for any Radon probability measure μ on K, and any $t \in [h_0(b(\mu)), h_1(b(\mu))]$, there exists a Radon probability measure ν on K_0, such that*

$$p_*(\nu) = \mu \qquad \textit{and} \qquad b(\nu) = (b(\mu), t).$$

Proof. If h_0 and h_1 are continuous, then K_0, $\Gamma(h_0)$, and $\Gamma(h_1)$ are compact convex sets, and $p_0 = p \mid \Gamma(h_0)$ and $p_1 = p \mid \Gamma(h_1)$ are affine isomorphisms and homeomorphisms of $\Gamma(h_0)$, resp. $\Gamma(h_1)$, onto K. Therefore, for any Radon probability measure μ on K there exists a unique Radon probability measure μ_i on $\Gamma(h_i)$, such that

$$(p_i)_* (\mu_i) = \mu, \ i = 0, 1.$$

For any $t \in [h_0(b(\mu)), h_1(b(\mu))]$, we have a unique $\lambda \in [0, 1]$, such that

$$t = (1 - \lambda)\, h_0(b(\mu)) + \lambda h_1(b(\mu)).$$

Since we can assume that μ_0 and μ_1 are Radon probability measures on the compact convex space K_0, we can consider the Radon probability measure

$$\nu = (1 - \lambda)\, \mu_0 + \lambda \mu_1,$$

on K_0, whose barycenter is $b(\nu) = (b(\mu), t)$. The lemma is proved.

We shall now extend the preceding Lemma to the case in which h_0 is assumed only to be lower semicontinuous, whereas h_1 is assumed only to be upper semicontinuous, but both are affine, and such that

$$h_0(x) \leqslant h_1(x), \qquad x \in K.$$

By a theorem of Krause, any semicontinuous affine real function on K is bounded (see [11, Satz 1]).

The set K_0, defined as in (IV.1), is again compact and convex, and we can consider the compact convex set $\mathscr{M}^1_+(K_0)$ of all Radon probability measures on K_0; by $\mathscr{M}^1_{(x,t)}(K_0)$ we shall denote the compact convex subset of $\mathscr{M}^1_+(K_0)$, consisting of all $\nu \in \mathscr{M}^1_+(K_0)$, such that $b(\nu) = (x, t)$, for $(x, t) \in K_0$.

PROPOSITION 1. *For any* $\mu \in \mathscr{M}^1_+(K)$ *and any* $t \in [h_0(b(\mu)), h_1(b(\mu))]$, *there exists a* $\nu \in \mathscr{M}^1_+(K_0)$, *such that*

$$p_*(\nu) = \mu \qquad \text{and} \quad b(\nu) = (b(\mu), t).$$

Proof. Since h_0 is affine and lower semicontinuous on K, there exists an increasing net $(h'_\alpha)_{\alpha \in A}$ of continuous affine functions $h'_\alpha \colon K \to \mathbb{R}$, such that

$$h_0(x) = \sup\{h'_\alpha(x); \alpha \in A\}, \qquad x \in K;$$

similarly, since h_1 is affine and upper semicontinuous on K, there exists a decreasing net $(h''_\alpha)_{\alpha \in A}$ of continuous affine functions $h''_\alpha \colon K \to \mathbb{R}$, such that

$$h_1(x) = \inf\{h''_\alpha(x); \alpha \in A\}, \qquad x \in K.$$

(Of course, we can assume that the directed set A is the same for the two nets.)

Let $K_\alpha = \{(x, t);\ x \in K,\ h'_\alpha(x) \leqslant t \leqslant h''_\alpha(x)\}$, $\alpha \in A$. Then $(K_\alpha)_{\alpha \in A}$ is a decreasing net of compact convex sets $K_\alpha \subset K \times \mathbb{R}$, and we have

$$\bigcap_{\alpha \in A} K_\alpha = K_0.$$

Since we have $K_\alpha \supset K_\beta$, for $\alpha \leqslant \beta$ in A, and

$$[h_0(x), h_1(x)] \subset [h'_\alpha(x), h''_\alpha(x)], \qquad \alpha \in A,$$

for any $x \in K$, we can consider that, for any $x \in K$ and any $t \in [h_0(x), h_1(x)]$, we have

$$\mathscr{M}^1_{(x,t)}(K_\alpha) \supset \mathscr{M}^1_{(x,t)}(K_\beta), \qquad \text{for} \quad \alpha \leqslant \beta \text{ in } A. \tag{IV.2}$$

Of course, we have that

$$\bigcap_{\alpha \in A} \mathscr{M}^1_{(x,t)}(K_\alpha) = \mathscr{M}^1_{(x,t)}(K_0). \tag{IV.3}$$

Since the sets in this equality are w^*-compact, whereas the mapping

$$p_* \colon \mathscr{M}^1_+(K_\alpha) \to \mathscr{M}^1_+(K)$$

is continuous, for any $\alpha \in A$, we have that

$$p_*\left(\bigcap_{\alpha \in A} \mathcal{M}^1_{(x,t)}(K_\alpha)\right) = \bigcap_{\alpha \in A} p_*(\mathcal{M}^1_{(x,t)}(K_\alpha)), \tag{IV.4}$$

where we have also taken into account relation (IV.2).

From (IV.3), (IV.4), and Lemma 2, we infer that

$$p_*(\mathcal{M}^1_{(x,t)}(K_0)) = \mathcal{M}^1_x(K), \qquad x \in K,\ t \in [h_0(x), h_1(x)],$$

and the Proposition is proved.

PROPOSITION 2. *For any maximal* $\mu \in \mathcal{M}^1_+(K)$ *and any* $t \in [h_0(b(\mu)), h_1(b(\mu))]$, *there exists a maximal* $\nu \in \mathcal{M}^1_+(K_0)$, *such that*

$$p_*(\nu) = \mu \qquad \text{and} \qquad b(\nu) = (b(\mu), t).$$

Proof. Assume that $\mu \in \mathcal{M}^1_+(K)$ is Choquet–Meyer maximal on K. By Proposition 1, there exists a Radon probability measure $\lambda \in \mathcal{M}^1_+(K_0)$, such that

$$p_*(\lambda) = \mu \qquad \text{and} \qquad b(\lambda) = (b(\mu), t).$$

Let $\nu \in \mathcal{M}^1_+(K_0)$ be a Choquet–Meyer maximal Radon probability measure on K_0, such that $\lambda \prec \nu$. Then we have that

$$\mu = p_*(\lambda) \prec p_*(\nu),$$

and, by the maximality of μ, we infer that

$$\mu = p_*(\nu).$$

Of course, $b(\nu) = b(\lambda) = (b(\mu), t)$. The Proposition is proved.

V

In this section we shall prove the following main result of the paper.

THEOREM 1. *For any semicontinuous affine function* $h_0\colon K \to \mathbb{R}$, *and any Choquet–Meyer maximal Radon probability measure* μ *on* K, *the function* $h_0 \mid \operatorname{ex} K$ *is* $\tilde{\mu}$*-measurable, and*

$$h_0(b(\mu)) = \int_{\operatorname{ex} K} h_0(x)\, d\tilde{\mu}(x). \tag{$*$}$$

Proof. (a) Without any loss of generality, we can assume that h_0 is lower semicontinuous and $0 \leqslant h_0(x) < 1$, for any $x \in K$. We shall consider the compact convex set

$$K_0 = \{(x, t); x \in K, h_0(x) \leqslant t \leqslant 1\} \subset E \times \mathbb{R},$$

corresponding to the functions h_0 and $h_1 = 1$, according to the preceding notation. From Proposition 2 we infer that there exists a Choquet–Meyer maximal Radon probability measure ν on K_0, such that

$$p_*(\nu) = \mu \qquad \text{and} \qquad b(\nu) = (b(\mu), h_0(b(\mu))).$$

The sets

$$K_n = \left\{(x, t) \in K_0; h_0(x) \leqslant t < h_0(x) + \frac{1}{n}\right\} \subset K_0, \qquad n \geqslant 1,$$

are obviously convex and open in K_0; since

$$\Gamma(h_0) = \bigcap_{n \geqslant 1} K_n,$$

we infer that $\Gamma(h_0)$ is a G_δ-subset, and a face, of K_0. On the other hand, the set

$$L = \{(x, 1); x \in K\}$$

is a compact face of K_0.

From Lemma 1 we infer that

$$\text{ex } L = \{(x, 1); x \in \text{ex } K\}$$

is a C-closed subset of ex K_0, and, therefore,

$$\text{ex } \Gamma(h_0) = \Gamma(h_0) \cap (\text{ex } K_0) = \{(x, h_0(x)); x \in \text{ex } K\}$$

is a C-open subset of ex K_0.

Since the function H_0: $K_0 \to \mathbb{R}$, given by

$$H_0(x, t) = t - h_0(x), \qquad (x, t) \in K_0,$$

is affine and upper semicontinuous on K_0, we have

$$0 = H_0(b(\mu), h_0(b(\mu))) = H_0(b(\nu)) = \int_{K_0} H_0(x, t)\, d\nu(x, t);$$

since $H_0 \geqslant 0$ on K_0, we infer that $H_0 = 0$, ν-a.e. on K_0, and this implies that

$$\nu(\Gamma(h_0)) = 1.$$

(b) Let now $a \in \mathbb{R}$, and define

$$E_a = \{x \in K; h_0(x) \leqslant a\}, \qquad \tilde{E}_a = E_a \cap (\text{ex } K),$$
$$F_a = \{x \in K; h_0(x) > a\}, \qquad \tilde{F}_a = F_a \cap (\text{ex } K);$$

E_a is a compact convex subset of K, whereas F_a is an open convex subset of K. We want to prove that $\tilde{E}_a$ (and $\tilde{F}_a = (\text{ex } K)\backslash\tilde{E}_a$) are $\tilde{\mu}$-measurable, for any $a \in \mathbb{R}$.

Indeed, since F_a is open in K, we have

$$\begin{aligned}\mu(F_a) &= \sup\{\mu(D); D \subset K \text{ compact, Baire, } D \subset F_a\}\\ &= \sup\{\tilde{\mu}_0(D \cap (\text{ex } K)); D \subset K \text{ compact, Baire, } D \subset F_a\}\\ &\leqslant (\tilde{\mu}_0)_* (\tilde{F}_a) \leqslant (\tilde{\mu})_* (\tilde{F}_a),\end{aligned} \tag{V.1}$$

where $\tilde{\mu}_0$ denotes the boundary measure induced by μ on $\tilde{\mathscr{B}}_0(\text{ex } K)$, as above, whereas $\tilde{\mu}$ denotes the boundary measure induced by μ on $\mathscr{B}(\text{ex } K; C)$. We still have to prove that

$$(\tilde{\mu})^* (\tilde{F}_a) \leqslant \mu(F_a), \qquad a \in \mathbb{R},$$

and this is equivalent to proving that

$$\mu(E_a) \leqslant (\tilde{\mu})_* (\tilde{E}_a), \qquad a \in \mathbb{R}.$$

For $a \geqslant 1$, the preceding inequality obviously holds. Therefore we can assume that $a < 1$.

Since the mapping q: $K_0 \ni (x, t) \mapsto q(x, t) = t$ is affine and continuous, we have that $q \mid \text{ex } K_0$ is $\tilde{\nu}$-measurable and

$$\begin{aligned}h_0(b(\mu)) = q(b(\nu)) &= \int_{\text{ex } K_0} t \, d\tilde{\nu}(x, t)\\ &= \int_{\text{ex } \Gamma(h_0)} t \, d\tilde{\nu}(x, t),\end{aligned}$$

where the second equality is a consequence of the fact that $\tilde{\nu}(\text{ex } \Gamma(h_0)) = 1$.

Since $\Gamma(h_0)$ is a G_δ-subset of K_0, for any $\varepsilon > 0$ there exists a compact extremal subset $D \subset K_0$, such that

$$D \subset \Gamma(h_0) \qquad \text{and} \qquad \nu(D) > 1 - \varepsilon$$

(see [18, Theorem 2]). Let us denote

$$G_a = \{(x, t) \in K_0; q(x, t) \leqslant a\},$$
$$\tilde{G}_a = G_a \cap (\mathrm{ex}\ K_0).$$

From $a < 1$, we infer that $\tilde{G}_a = G_a \cap (\mathrm{ex}\ \Gamma(h_0))$, and we have that

$$E_a = p(G_a),\ \tilde{E}_a = p(\tilde{G}_a).$$

Since G_a is Baire measurable in K_0, the set $\tilde{G}_a$ is $\tilde{\nu}$-measurable; by the regularity of $\tilde{\nu}$, we infer that there exists a compact extremal subset $D_1 \subset K_0$, such that

$$D_1 \cap (\mathrm{ex}\ K_0) \subset \tilde{G}_a$$

and

$$\nu(D_1) = \tilde{\nu}(D_1 \cap (\mathrm{ex}\ K_0)) > \tilde{\nu}(G_a) - \varepsilon.$$

We infer that the set $D_0 = D \cap D_1$ is compact extremal in K_0, and we have

$$D_0 \subset \Gamma(h_0) \qquad \text{and} \qquad \nu(D_0) > \tilde{\nu}(G_a) - 2\varepsilon.$$

Since $p \mid \Gamma(h_0)$ is an affine isomorphism of $\Gamma(h_0)$ onto K, and since it is continuous, the set $p(D_0) \subset K$ is compact and extremal in K, and

$$\begin{aligned}\mu(p(D_0)) &= \nu(p^{-1}(p(D_0))) = \nu(p^{-1}(p(D_0)) \cap \Gamma(h_0)) \\ &= \nu(D_0) > \tilde{\nu}(G_a) - 2\varepsilon.\end{aligned}$$

Since we have that

$$p(D_0) \cap (\mathrm{ex}\ K) \subset \tilde{E}_a,$$

we infer that

$$(\tilde{\mu})_* (\tilde{E}_a) \geqslant \tilde{\mu}(p(D_0) \cap (\mathrm{ex}\ K)) = \mu(p(D_0)) > \tilde{\nu}(\tilde{G}_a) - 2\varepsilon. \tag{V.2}$$

Since G_a is Baire measurable in K_0, we have

$$\begin{aligned}\tilde{\nu}(\tilde{G}_a) &= \nu(G_a) = \nu(G_a \cap \Gamma(h_0)) \leqslant \nu(p^{-1}(p(G_a))) \\ &= \nu(p^{-1}(p(G_a)) \cap \Gamma(h_0)) = \nu(G_a \cap \Gamma(h_0)),\end{aligned}$$

and this implies that

$$\mu(E_a) = \nu(p^{-1}(E_a)) = \nu(p^{-1}(p(G_a))) = \tilde{\nu}(\tilde{G}_a). \tag{V.3}$$

From (V.2) and (V.3) we infer that

$$(\tilde{\mu})_*(\tilde{E}_a) \geqslant \mu(E_a), \tag{V.4}$$

and the $\tilde{\mu}$-measurability of the sets $\tilde{E}_a$, $a \in \mathbb{R}$, is proved.

(c) From (V.1) and (V.4) we infer that

$$\tilde{\mu}(\tilde{E}_a) = \mu(E_a), \qquad \forall a \in \mathbb{R}. \tag{V.5}$$

Since the barycentric calculus holds for h_0, we have that

$$\begin{aligned} h_0(b(\mu)) &= \int_K h_0(x)\, d\mu(x) = \int_{\mathbb{R}} a d\mu(E_a) \\ &= \int_{\mathbb{R}} a d\tilde{\mu}(\tilde{E}_a) = \int_{\operatorname{ex} K} h_0(x)\, d\tilde{\mu}(x), \end{aligned}$$

where we have used (V.5) and the Riemann–Stieltjes integral. The Theorem is proved.

VI

We shall now extend to the general case of any compact convex set the notion of a *univerally measurable element*, which was introduced by G. K. Pedersen for the case of the C^*-algebras (see [13, p. 104; 14]).

Namely, we shall say that an affine function φ: $K \to \mathbb{R}$ is *strongly universally measurable* if for any $x \in K$ and any $\varepsilon > 0$, there exist lower semicontinuous affine functions h, k: $K \to \mathbb{R}$, such that

$$-k \leqslant \varphi \leqslant h \qquad \text{and} \qquad h(x) + k(x) < \varepsilon.$$

It is obvious that any semicontinuous affine function on K is strongly universally measurable; and also, that the set of all strongly universally measurable affine real functions on K is a Banach subspace of the space $A^b(K)$ of all bounded affine real functions on K, equipped with the sup-norm.

We shall prove now the following

THEOREM 2. *For any strongly universally measurable affine function* φ: $K \to \mathbb{R}$, *the following properties hold*:

(a) φ *is universally measurable* (*with respect to any Radon measure on* K).

(b) $\varphi \mid \operatorname{ex} K$ *is* $\tilde{\mu}$*-measurable, for any measure* $\tilde{\mu}$ *induced on* $\mathscr{B}(\operatorname{ex} K; C)$ *by any Choquet–Meyer maximal Radon probability measure* μ *on* K.

(c) *The barycentric calculus holds for* φ; *i.e., we have that*

$$\varphi(b(\mu)) = \int_K \varphi(x)\, d\mu(x),$$

for any Radon probability measure μ *on* K.

(d) *The boundary barycentric calculus holds for* φ; *i.e., we have that*

$$\varphi(b(\mu)) = \int_{\text{ex } K} \varphi(x)\, d\tilde{\mu}(x),$$

for any Choquet–Meyer maximal Radon probability measure μ *on* K, *and the corresponding boundary measure* $\tilde{\mu}$, *induced on* $\mathscr{B}(\text{ex } K; C)$, *by* μ.

Proof. (a) Let μ be any Radon probability measure on K, and denote $x_0 = b(\mu)$. Then, for any $\varepsilon > 0$, there exist lower semicontinuous affine functions h_ε, k_ε: $K \to \mathbb{R}$, such that

$$-k_\varepsilon \leqslant \varphi \leqslant h_\varepsilon \qquad \text{and} \qquad h_\varepsilon(x_0) + k_\varepsilon(x_0) < \varepsilon.$$

Let then a_n: $K \to \mathbb{R}$ and b_n: $K \to \mathbb{R}$, $n \geqslant 1$, be defined by

$$a_n(x) = \max\{-k_1(x), -k_{1/2}(x), ..., -k_{1/n}(x)\},$$
$$b_n(x) = \min\{h_1(x), h_{1/2}(x), ..., h_{1/n}(x)\},$$

for any $x \in K$, and define a: $K \to \mathbb{R}$ and b: $K \to \mathbb{R}$ by

$$a(x) = \sup\{a_n(x); n \geqslant 1\}, \qquad x \in K,$$
$$b(x) = \inf\{b_n(x); n \geqslant 1\}, \qquad x \in K.$$

It is obvious that a and b are Borel measurable on K, that

$$a(x) \leqslant \varphi(x) \leqslant b(x), \qquad x \in K, \tag{VI.1}$$

and, since the barycentric calculus holds for semicontinuous affine functions, we have

$$\begin{aligned} 0 &\leqslant \int_K (b(x) - a(x))\, d\mu(x) = \lim_{n \to \infty} \int_K (b_n(x) - a_n(x))\, d\mu(x) \\ &\leqslant \inf\left\{\int_K (h_{1/n}(x) + k_{1/n}(x))\, d\mu(x); n \geqslant 1\right\} \\ &= \inf\{h_{1/n}(x_0) + k_{1/n}(x_0); n \geqslant 1\} = 0. \end{aligned} \tag{VI.2}$$

From (VI.1) and (VI.2) we infer that φ is μ-measurable.

(c) It is easy to infer that we have

$$\varphi(b(\mu)) = \int_K \varphi(x)\, d\mu(x),$$

for any Radon probability measure μ on K.

(b) and (d) have a similar proof, with the help of Theorem 1. The Theorem is proved.

VII

The preceding results enable us to give a partial answer to a question raised by Goullet de Rugy (see [6, p. 142; 10]). Namely, we have the following

THEOREM 3. *If φ, ψ: $K \to \mathbb{R}$ are strongly universally measurable affine functions, such that $\varphi \leqslant \psi$ on* ex K, *then $\varphi \leqslant \psi$ on K.*

Proof. It is an immediate consequence of the boundary barycentric calculus, but it can also be proved directly.

REMARKS

1. If K is the state space $S(C(X))$ of the commutative C^*-algebra $C(X)$ of all continuous complex functions on the compact space X, then, for any Choquet–Meyer maximal Radon probability measure μ on K, $\tilde{\mu}$ is the Borel–Radon probability measure, corresponding to the state $b(\mu) \in S(C(X))$, provided X is identified with ex $S(C(X))$, by the mapping $X \ni x \mapsto \varepsilon_x \in S(C(X))$; whereas $\tilde{\mu}_0$ is only the Baire–Radon probability measure on X, corresponding to the state $b(\mu)$.

2. Keeping the same notations as above, let $\tilde{\varphi}$: $X \to \mathbb{R}$ be a bounded (lower) semicontinuous real function. The usual method of extending a (normed) Radon integral $I \in S(C(X))$ to a larger class of functions starts with the definition of $I(\varphi)$, which yields a mapping

$$S(C(X)) \ni I \mapsto I(\varphi) \in \mathbb{R},$$

which is manifestly (lower) semicontinuous and affine. Theorem 1 says, in this case, that any (lower) semicontinuous affine function on $S(C(X))$ arises in this way, but this is obvious, in view of the fact that $S(C(X))$ is a Bauer simplex. For a discussion referring to Choquet simplexes, see [21].

3. Question 1 from [10] is to be answered by *no*, even for the case of $K = S(C(X))$, where X is any compact space possessing Borel measurable (compact) subsets, which do not belong to the Lebesgue completion of the σ-algebra of the Baire measurable subsets, with respect to (at least) one Radon probability measure. Therefore, this question should be replaced by

Question 1′. If $\varphi\colon K \to \mathbb{R}$ is any universally measurable affine function, satisfying the barycentric calculus, i.e., such that

$$\varphi(b(\mu)) = \int_K \varphi(x)\, d\mu(x), \qquad \forall \mu \in \mathcal{M}^1_+(K),$$

does φ satisfy the *boundary* barycentric calculus? That is, does it follow that $\varphi \mid \operatorname{ex} K$ is $\tilde{\mu}$-measurable, for the measure induced on $\mathcal{B}(\operatorname{ex} K; C)$ by *any* Choquet–Meyer maximal Radon probability measure μ on K, and does the equality

$$\varphi(b(\mu)) = \int_{\operatorname{ex} K} \varphi(x)\, d\tilde{\mu}(x)$$

hold?

4. The present paper is a slightly improved version of the preprint [20], which had a limited distribution.

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